Steady flows near the critical speed

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1. Introduction

The solitary wave is a well-known example of a flow of permanent type which can be studied by shallow-water theory. A scheme due to K. O. Friedrichs (see Friedrichs & Hyers 1954) can give both an approximation to the exact wave form and a framework for a proof of its existence. It is used here to approximate the solution of a related problem, that of a steady flow near the critical speed over a small obstacle in the bed of a stream. The flow can be considered as a perturbation of a solitary wave since it will approach a solitary wave as the obstacle shrinks in height.

A similar problem, that of the motion of a vortex under the surface of a fluid. was solved recently by Filippov (1960). He used the Friedrichs scheme to obtain an approximate solution, and a modification of the Friedrichs–Hyers argument to prove existence. His solution approaches the solitary wave as the vortex strength diminishes.

In the present problem, the bottom is assumed to be flat up to the point at which the obstacle occurs. Since a flow of permanent type is being studied, the water can be assumed to be moving under a fixed surface. This surface is approximated by a solitary wave upstream. Downstream, the approximating surface is in general periodic. However, for each sufficiently small obstacle, there is an exceptional flow which is non-periodic downstream. This flow can be described as two solitary waves pieced together with a distortion in the neighbourhood of the obstacle. It is referred to here as a piece-wise solitary wave.

By a preliminary normalization, the velocity of the flow far upstream (at $x = -\infty$) is unity. In addition, the depth of the water there is the unit of length. The dimensionless variable $z = x+iy$ is defined relative to a co-ordinate system with the x-axis parallel to the flat portion of the bottom, the y-axis oriented toward the surface, and the origin at the point at which the flow first meets the obstacle. The complex velocity potential is denoted by $\chi = \phi + i\psi$. The free surface corresponds to $\psi = 1$, the bottom to $\psi = 0$. At the free surface we have the Bernoulli boundary condition

$$\tfrac{1}{2}|w|^2+\gamma y = \text{const.} \quad \text{at} \quad \psi = 1, \tag{1.1}$$

where $\gamma = gh/U^2$, g is the acceleration of gravity, h is the depth far upstream before normalization, U is the velocity far upstream before normalization, and w is the complex velocity $u-iv$. All flows considered here are supercritical, that is, γ is less than unity. At the bottom, the normal component of the velocity is zero. Upstream, we have the condition

$$w \to 1 \quad \text{as} \quad x \to -\infty. \tag{1.2}$$

The downstream condition will depend on the obstacle and will be discussed later.

It is convenient to replace the variable w by the pair of real harmonic functions θ and τ defined by

$$w = \exp[-i\{\theta + i(\tau - a^2)\}], \tag{1.3}$$

where $$|w| = \exp(\tau - a^2), \quad \theta = -\arg w, \quad \gamma = \exp(-3a^2).$$

The functions θ and τ are to be solutions to the Cauchy–Riemann equations in the strip $0 < \psi < 1$, $-\infty < \phi < +\infty$. It is required that (from (1·2))

$$\theta \to 0, \quad \tau \to a \quad \text{as} \quad \phi \to -\infty. \tag{1.4}$$

The surface condition is

$$\partial\theta/\partial\psi = \exp(-3\tau)\sin\theta \quad \text{at} \quad \psi = 1. \tag{1.5}$$

This is derived by differentiating the Bernoulli equation (1.1) with respect to ϕ (see ch. 10 of Stoker 1957).

If θ and τ exist and are suitably bounded, then the potential function $\chi(z)$ maps the portion of the physical plane (the z-plane) containing the flow simply and conformally onto the strip bounded by $\psi = 0$ and $\psi = 1$ in the χ-plane. The inverse mapping function

$$z(\chi) = \int_0^{\chi} \frac{d\chi}{w(\chi)} \tag{1.6}$$

can be used once we have solved for the velocity function $w(\chi)$ (see ch. 12 of Stoker 1957).

The normal component of the velocity must vanish along the bottom. Therefore if $y = Q(x)$ is the equation of the bottom, then

$$\tan\theta\{x, Q(x)\} = dQ(x)/dx \quad (-\infty < x < \infty). \tag{1.7}$$

Although this condition does not easily transform to the χ-plane, the following iterative procedure can be used: first take as a boundary condition

$$\theta^0(\phi, 0) = f^0(\phi) = \tan^{-1}[dQ(\phi)/dx]; \tag{1.8}$$

solve the boundary value problem to get

$$x'(\phi, 0) = \int_0^{\phi} \exp[-\tau^0(\phi, 0)]\cos\theta^0(\phi, 0)\,d\phi; \tag{1.9}$$

now solve the problem using

$$\theta' = f'(\phi) = \tan^{-1} dQ[x'(\phi, 0)]/dx, \tag{1.10}$$

etc. The proof of the convergence of this procedure requires knowledge of the dependence of τ on the boundary-function f. Within the framework of an approximate theory, the following can be stated. If the expansion of τ and θ given in the next section is truncated, and if the resulting approximations to τ and θ are used in the iterative procedure, then convergence can be demonstrated. This is outlined in § 5.

2. The expansion procedure

The Friedrichs procedure is to stretch the horizontal variable and expand the unknowns τ and θ in powers of the stretching parameter. In the present problem the number γ is less than but close to unity, i.e. the supercritical flow is near the critical speed. The stretching parameter is taken to be

$$a = (-\tfrac{1}{3}\log\gamma)^{\frac{1}{2}}. \tag{2.1}$$

Let

$$\overline{\phi} = a\phi, \quad \overline{\psi} = \psi,$$

$$\tau(\overline{\phi}/a, \overline{\psi}) = \sum_{n=1}^{\infty} a^{2n}\tau_{2n}(\overline{\phi}, \overline{\psi}),$$

$$\theta(\overline{\phi}/a, \overline{\psi}) = \sum_{n=1}^{\infty} a^{2n+1}\theta_{2n+1}(\overline{\phi}, \overline{\psi}) \tag{2.2}$$

(omitting terms which can be shown to be zero). The function f is assumed to have the expansion

$$f(\overline{\phi}/a) = \sum_{n=2}^{\infty} a^{2n+1}f_{2n+1}(\overline{\phi}). \tag{2.3}$$

(In the following, the '$-$' will be omitted.)

The Cauchy–Riemann equations become

$$\partial\theta/\partial\psi = -a\,\partial\tau/\partial\phi, \quad \partial\tau/\partial\psi = a\,\partial\theta/\partial\phi. \tag{2.4}$$

Hence

$$\sum_{n=1}^{\infty} a^{2n+1}\left(\frac{\partial\theta_{2n+1}}{\partial\psi} + \frac{\partial\tau_{2n}}{\partial\phi}\right) = 0, \quad a^2\frac{\partial\tau_2}{\partial\psi} + \sum_{n=2}^{\infty} a^{2n}\left(\frac{\partial\tau_{2n}}{\partial\psi} - \frac{\partial\theta_{2n-1}}{\partial\phi}\right) = 0, \tag{2.5}$$

or

$$\frac{\partial\tau_2}{\partial\psi} = 0, \quad \frac{\partial\theta_{2n+1}}{\partial\psi} = -\frac{\partial\tau_{2n}}{\partial\phi}, \quad \frac{\partial\tau_{2n+2}}{\partial\psi} = \frac{\partial\theta_{2n+1}}{\partial\phi} \quad (n \geqslant 1). \tag{2.6}$$

At $\psi = 0$, we have from (2.3) and (1.8) that

$$\theta_{2n+1} = f_{2n+1}. \tag{2.7}$$

Let

$$\tau_{2n}(\phi, 0) = j_{2n}(\phi). \tag{2.8}$$

If the j_{2n} are known, then it follows from (2.6) that θ_{2n+1} and τ_{2n} are completely determined. For example

$$\left.\begin{aligned} &\tau_2 = j_2, \quad \theta_3 = -\psi j_2', \\ &\tau_4 = -\tfrac{1}{2}\psi^2 j_2'' + j_4, \quad \theta_5 = (\psi^3/3!)f_2''' - \psi j_4' + f_5, \\ &\tau_6 = (\psi^4/4!)j_2^{(4)} - \tfrac{1}{2}\psi^2 j_4'' + \psi f_5' + j_6, \\ &\theta_7 = -(\psi^5/5!)j_2^{(5)} + (\psi^3/3!)j_4''' - \tfrac{1}{2}\psi^2 f_2'' - \psi j_6' + f_7. \end{aligned}\right\} \tag{2.9}$$

The ordinary differential equations which are satisfied by the j_{2n} are found by inserting the expansions for τ and θ into the surface condition (1.6). It can be written as

$$-a\exp(3\tau)\,\partial\tau/\partial\phi = \sin\theta \quad (\psi = 1), \tag{2.10}$$

or (with $\tau' = \partial\tau/\partial\phi$ at $\psi = 1$)

$$\tau' + 3\tau\tau' + \theta/a = G = \sum_{n=3}^{\infty} a^{2n}g_{2n}(\phi), \tag{2.11}$$

where

$$G = (\sin\theta - \theta)/a + \tau'[\exp(3\tau) - 1 - 3\tau].$$

It follows that at $\psi = 1$,

$$\left.\begin{aligned} &\tau_2' + \theta_3 = 0, \\ &\tau_4' + \theta_5 + 3\tau_2\tau_2' = 0, \\ &\tau_6' + \theta_7 + 3(\tau_4\tau_2' + \tau_2\tau_4') = -\tfrac{3}{2}\tau_2\tau_2', \\ &\tau_{2n}' + \theta_{2n+1} + 3\sum_{k=1}^{n-1} \tau_{2(n-k)}\tau_{2k}' = g_{2n} \quad (n \geqslant 3). \end{aligned}\right\} \tag{2.12}$$

It is important that g_{2n} is formed from the τ_{2k}, τ_{2k}', and θ_{2k+1}, where $1 \leqslant k \leqslant n-2$.

We now have

$$\tau_2 = j_2, \quad \theta_3 = -\psi j_2'. \tag{2.13}$$

Since at $\psi = 1$,

$$\tau_4' + \theta_5 = -\tfrac{1}{2}j_2''' = j_4' + \tfrac{1}{6}j_2''' - j_4' + f_5, \tag{2.14}$$

we have

$$j_2''' = 9j_2 j_4' + 3f_5, \tag{2.15}$$

the differential equation which determines the dominant term in the expansion of τ.

We also have

$$\tau_6' + \theta_7 = \tfrac{1}{3}j_2^{(5)} + \tfrac{1}{2}f_5'' + f_7, \tag{2.16}$$

or

$$j_4''' = 9j_2 j_4' + h_4, \tag{2.17}$$

where

$$h_4 = f_7 + \tfrac{3}{2}f_5'' + \tfrac{9}{2}j_2 j_2' - \tfrac{9}{2}j_2 j_2''' - \tfrac{9}{2}j_2' j_2'' + \tfrac{1}{10}j_2^{(5)}.$$

These calculations were carried through in detail to illustrate the fact (which can be proved by induction) that in the expression $\tau_{2n+2}' + \theta_{2n+3}$, the j_{2n+2}' term cancels. We therefore obtain a linear differential equation for j_{2n}, namely

$$j_{2n}''' = 9j_2 j_{2n}' + h_{2n} \quad (n \geqslant 2), \tag{2.18}$$

where h_{2n} is a polynomial in $j_{2k}^{(m)}$ $(k < n)$ and the derivatives of the known functions f_{2m+1}.

In the next two sections, the equations for j_2 and j_{2n} will be studied. It will be shown that the presence of the obstacle determines the character of j_2. Using j_2, a function class is defined, and it is shown that the linear differential equations for j_{2n} $(n > 1)$, have unique solutions in this class. This implies that the terms in the formal expansions for θ and τ can be calculated.

3. The dominant term, j_2

The function j_2 satisfies the differential equation (omitting subscripts)

$$j''' = 9jj' + 3f, \tag{3.1}$$

with the boundary conditions

$$j(-\infty) = 1, \quad j'(-\infty) = j''(-\infty) = 0. \tag{3.2}$$

The function $f = dF/d\phi$, with $F = 0$ outside the interval $0 \leqslant \phi \leqslant A$. We integrate (3.1) to obtain

$$j'' = \tfrac{9}{2}(j^2 - 1) + 3F. \tag{3.3}$$

It is helpful to interpret the differential equation (3.3) as one governing the motion of a particle in a position-dependent force field subject to a time-dependent perturbation. For this purpose, j is taken as the position of the particle and ϕ as the time. Equation (3.3) can be written as the derivative of the energy, E,

$$E' = [j'^2 + V(j)]' = 6j'F, \tag{3.4}$$

where the potential energy function is given by

$$V = 9(j - \tfrac{1}{3}j^3). \tag{3.5}$$

Clearly
$$E = j'^2 + V(j) = 6\left[1 + \int_{-\infty}^{\phi} j'(s)\,F(s)\,ds\right]. \tag{3.6}$$

The conservative part of the force which is acting on the particle is derivable from the potential function V. The time rate of change of energy is equal to the product of the non-conservative part of the force and the velocity, j'. If the non-conservative part of the force opposes the motion, the energy of the system decreases. It is this decrease in energy which changes the solitary wave into a periodic one.

If F is identically zero, the energy is equal to six. The motion of the particle is described by a bounded non-periodic solution of (3.3). The particle moves from the left, $j = 1$ at $\phi = -\infty$, and reverses direction at $j = -2$, where j' vanishes. It then returns to $j = 1$. This solution corresponds to the solitary wave with j' vanishing at the crest.

Even with the obstacle present, F is zero for ϕ negative. Therefore j is given by the solitary wave solution on $(-\infty, 0)$, i.e.

$$\left.\begin{aligned} j(\phi) &= 1 - 3\,\mathrm{sech}^2\,(\tfrac{3}{2}\phi - b) \quad (\phi < 0),\\ j'(\phi) &= 9\,\mathrm{sech}^2\,(\tfrac{3}{2}\phi - b)\tanh\,(\tfrac{3}{2}\phi - b).\end{aligned}\right\} \tag{3.7}$$

If b is positive, the particle is moving to the left when it encounters the perturbation. To oppose the motion, F must be positive. If b is negative, the particle has reversed its direction: F must therefore be negative. In terms of the flow, b positive means the crest has not formed before the wave encounters the obstacle. To obtain periodic motion, the obstacle must be a bump, i.e. F positive. If b is negative, the crest has been formed, and a dip is needed for periodicity. We will suppose that b is positive; the analysis for b negative is similar.

We therefore assume that F is non-negative with support† in $[0, A]$. It will be shown that for each positive b, there exist bounds $C(b)$ and $A(b)$ such that if the maximum value of F is less than $C(b)$ and if F vanishes for ϕ greater than $A(b)$, then $j(\phi)$ is periodic for $\phi > A(b)$. A function F satisfying the above conditions is said to be *admissible* with respect to b.

The restrictions on F are conveniently described in terms of the integrals

$$\left.\begin{aligned} L &= \int_0^A F(s)\,ds, & M &= \int_0^A \int_0^t F(s)\,ds\,dt,\\ N &= \int_0^A sF(s)\,ds, & P &= \int_0^A \int_0^t F(t)\,F(s)\,ds\,dt.\end{aligned}\right\} \tag{3.8}$$

Since b is positive, we have

$$-\sqrt{12} \leqslant j'(0) < 0, \quad -2 \leqslant j(0) < \delta(b) \leqslant 1. \tag{3.9}$$

The bound δ can be taken as unity for the first part of the analysis. For the piecewise solitary wave, a sharper restriction is needed.

† A function is said to have its support in an interval if it is zero outside the interval.

It will be shown that A and C can be selected so that

$$j(0)+j'(0)A+\tfrac{27}{2}A^2+3M < \delta, \quad -2+\tfrac{27}{2}A^2+3M < \delta. \tag{3.10}$$

Condition (3.10) implies that $-2 \leqslant j(A) < \delta$, one of the necessary conditions for j to be bounded for $\phi > A$.

Proposition 1. If (3.10) is satisfied for $b = b_0$, then $-2 \leqslant j(\phi) < \delta$ for $0 \leqslant \phi \leqslant A$, and $0 \leqslant b \leqslant b_0$.

Proof. We first note that if $j^2 \leqslant 4$, then $-2 \leqslant j < \delta$. This is proved by integrating the differential equation (3.3) twice, using the bound on j, and the first inequality of (3.10).

Now let $0 \leqslant \phi \leqslant A_1 \leqslant A$ be the largest interval on which $-2 \leqslant j(\phi) \leqslant \delta$. It will be shown that $A_1 = A$. By the preceding paragraph, we know that j cannot equal δ. Therefore if $A_1 < A, j'(A_1) \leqslant 0$. But this implies that $j(A_1) = -2$ and $j''(A_1) > 0$. Hence $j'(A_1) < 0$. From the energy relation (3.6) we get that

$$\int_0^{A_1} j'(s)F(s)\,ds > 0. \tag{3.11}$$

Since $j'(0) < 0$, there must be an interval $[A_2, A_1]$ on which $j' \leqslant 0$ and $j'(A_2) = 0$, $-2 \leqslant j(A_2) < \delta$. Therefore $V\{j(A_2)\} < 6$, and

$$\int_0^{A_2} j'(s)F(s)\,ds < 0, \quad \int_{A_2}^{A_1} j'(s)F(s)\,ds \leqslant 0. \tag{3.12}$$

This contradicts (3.11).

We have shown that $-2 \leqslant j(\phi) < \delta$ for $0 \leqslant \phi \leqslant A$. That this bound holds uniformly in b can be shown by substituting the formulas for $j(0)$ and $j'(0)$ into (3.10). The resulting function is a cubic in $\tanh b$ whose range is bounded above by δ.

The bounds on A and C implied by (3.10) do not insure that $|E| \leqslant 6$, for $\phi \geqslant A$. In fact if we take $b = 0$, then if F is not identically zero, $E > 6$ after the perturbation. This follows from the fact that if $b = 0$, then $j'(\phi) > 0$ for $\phi > 0$. If this were not so, then at the first zero of j', $V > 6$, and therefore the corresponding value of j is less than -2; a contradiction.

To control the size of E, we assume that

$$-2 \leqslant j'(0)L+3P-\tfrac{9}{2}N \leqslant j'(0)L+\tfrac{27}{2}N < 0. \tag{3.13}$$

Proposition 2. If (3.13) is satisfied and $j^2(\phi) \leqslant 4$, then

$$-2 \leqslant \int_0^A j'(s)F(s)\,ds < 0.$$

Proof.

$$j'(\phi) = j'(0)+\frac{9}{2}\int_0^\phi [j(s^2)-1]\,ds+3\int_0^\phi F(s)\,ds. \tag{3.14}$$

Therefore

$$\int_0^A j'(s)F(s)\,ds = j'(0)\int_0^A F(s)\,ds+3\int_0^A\int_0^t F(t)F(s)\,ds\,dt + \frac{9}{2}\int_0^A\int_0^t F(t)[j(s)^2-1]\,ds\,dt. \tag{3.15}$$

For the lower bound, replace j^2 by 0; for the upper bound, replace j^2 by 4. We can now use the inequality (3.13).

It has been shown that if (3.10) and (3.13) are satisfied, then

$$-2 \leqslant j(A) < \delta \quad \text{and} \quad -6 \leqslant E < 6. \tag{3.16}$$

We conclude that for $\phi > A$, $j(\phi)$ is a periodic solution of

$$j'' = \tfrac{9}{2}(j^2-1). \tag{3.17}$$

It may be represented by the elliptic integral

$$\phi - A = \int_{j(A)}^{j(\phi)} [E - V(j)]^{-\frac{1}{2}}\, dj. \tag{3.18}$$

Its period is given by

$$T(b) = 2\int_{p}^{q} [E - V(j)]^{-\frac{1}{2}}\, dj, \tag{3.19}$$

where p and q are the points of intersection of $V = E$ and $V = V(j)$. We have $0 \leqslant F < C$, $0 \leqslant \phi \leqslant A$. Let $B = AC$, $c = j(0)$, $k = j'(0)$. Then

$$0 \leqslant L \leqslant B, \quad 0 \leqslant M \leqslant \tfrac{1}{2}AB, \quad 0 \leqslant N \leqslant \tfrac{1}{2}AB, \quad 0 \leqslant P \leqslant \tfrac{1}{2}B^2. \tag{3.20}$$

Proposition 3. If $-2 \leqslant c < \delta$, $-\sqrt{12} \leqslant k < 0$, and if A and B satisfy

$$\sqrt{12}B + \tfrac{9}{4}AB < 2, \tag{i}$$

$$k + \tfrac{3}{2}B + \tfrac{27}{4}A < 0, \tag{ii}$$

$$c + \tfrac{27}{4}A^2 < \delta, \tag{iii}$$

$$\tfrac{27}{2}A^2 + \tfrac{3}{2}AB < \delta + 2, \tag{iv}$$

then inequalities (3.10) and (3.13) are satisfied.

Proof. If (i), then

$$-2 < \sqrt{12}B - \tfrac{9}{4}AB \leqslant kL - \tfrac{9}{2}N \leqslant kL + 3P - \tfrac{9}{2}N. \tag{3.21}$$

If (ii), then

$$kL + 3P + \tfrac{27}{2}N \leqslant kB + \tfrac{3}{2}B + \tfrac{27}{4}AB < 0. \tag{3.22}$$

If (ii) and (iii), then

$$c + kA + \tfrac{27}{2}A^2 + 3M \leqslant c + A(k + \tfrac{27}{2}A + \tfrac{3}{2}B) \leqslant c + \tfrac{27}{2}A < \delta. \tag{3.23}$$

If (iv), then

$$\tfrac{27}{2}A^2 + 3M < \delta + 2. \tag{3.24}$$

If $c < \delta$, and $k < 0$, the inequalities (i)–(iv), $0 < A$, $0 < B$, define a non-empty region in the (A, B)-plane. All the bounds can therefore be satisfied, and we have

Theorem 1. If b is positive, then there exist constants $A(b)$ and $C(b)$ such that for any continuous function F with support in $[0, A(b)]$, such that $0 \leqslant F(\phi) < C(b)$, there is a unique solution of (3.3) which is bounded and non-periodic for $\phi < 0$, and periodic for $\phi > A$.

A piecewise solitary wave will occur if

$$-2 \leqslant j(A) < 1 \quad \text{and} \quad \int_0^A j'(s)F(s)\, ds = 0. \tag{3.25}$$

This effect can be produced by holding the parameter b fixed and increasing the perturbing function until (3.25) is satisfied. An alternative, which is used here, is to hold the perturbing function fixed and reduce b to the critical value. Both approaches seem to require that the perturbation take place near the crest. This will insure that the energy is a monotonic decreasing function of b in the neighbourhood of the critical value.

Theorem 2. If $b_0 < \tanh^{-1}(\sqrt{3}/3)$ and if F is any non-zero function which is admissible with respect to b_0, then there is a unique positive value of b less than b_0 such that (3.25) is satisfied. Therefore

$$j(\phi) = 1-3\,\mathrm{sech}^2(\tfrac{3}{2}\phi+g)\ (\phi > A), \tag{3.26}$$

where $\frac{3}{2}A+g > 0$. The flow is symmetric with respect to its crest if and only if F is symmetric.

Proof. Let

$$D(\phi, b) = \partial j/\partial b. \tag{3.27}$$

$$\left.\begin{aligned} D'' = 9jD, \quad \text{with} \quad D(0,b) &= 6\tanh b - 6\tanh^3 b, \\ D'(0,b) &= -9+36\tanh^2 b - 27\tanh^4 b. \end{aligned}\right\} \tag{3.28}$$

It is easy to show that if $-2 \leqslant j < -1$, then

$$D'(\phi,b) \leqslant D'(0,b)\{1-9A^2\} \quad (0 \leqslant \phi \leqslant A). \tag{3.29}$$

Take $\bar{b}$ so that $\tanh^2\bar{b} = \frac{1}{3}$. Then $j(0,\bar{b}) = -1$, $D'(0,\bar{b}) = 0$. So if we take the bound $\delta = -1$ in (3.10), we can conclude that if $0 \leqslant b < \bar{b}$, $-2 \leqslant j(\phi, b) < -1$ for $0 \leqslant \phi \leqslant A$. Since

$$\frac{\partial E}{\partial b} = 6\int_0^A D'(s,b)\,F(s)\,ds < 0, \tag{3.30}$$

$E(b)$ is monotonic decreasing.

If $b_0 < \bar{b}$, and F is admissible with respect to b_0, then $E(b_0) < 0$. Since $E(b)$ is a continuous monotonic decreasing function, and since $E(0) > 0$, there is a unique positive value of b such that $E(b) = 0$. For this value of b, (3.25) is satisfied.

The inequality $\frac{3}{2}A+g > 0$ is equivalent to $j'(A) > 0$. This second inequality follows from the fact that j' has only one zero during the perturbation.

In order to prove that the critical flow is symmetric if and only if the bump is symmetric, it is convenient to change the perturbation interval to $[-\frac{1}{2}A, \frac{1}{2}A]$. If F is not an even function, then from the differential equation (3.3), it follows that j cannot be an even function. Now let us assume that F is even. The condition that $E = 6$ can be written as

$$0 = \int_{-\frac{1}{2}A}^{\frac{1}{2}A} j'(s)\,F(s)\,ds = \int_0^{\frac{1}{2}A} [j'(s)+j'(-s)]\,F(s)\,ds. \tag{3.31}$$

Let $W(s) = j(s)-j(-s)$. Then

$$W'' = \tfrac{9}{2}[j(s)+j(-s)]\,W, \quad W(0) = 0, \quad W'(0) = 2j'(0). \tag{3.32}$$

Clearly j is even if and only if $j'(0) = 0$ (recall that $s = 0$ at $\phi = \frac{1}{2}A$).

Since $-2 \leqslant j$,

$$W''+18 = K(s)\,W, \quad \text{where} \quad 0 \leqslant K(s) = \tfrac{9}{2}[j(s)+j(-s)+4]. \tag{3.33}$$

We write

$$W'(s) = \int_0^s \cos r(s-t)\,K(t)\,W(t)\,dt + W'(0)\sin rs \quad (r^2 = 18). \tag{3.34}$$

If $rA < \pi$, then $W'(0) > 0$ implies that W and W' are positive on $[0, \frac{1}{2}A]$. This contradicts (3.32). If $W'(0) < 0$, we have a similar contradiction. Therefore $W'(0) = 0$, and j is an even function. Finally, we note that $rA < \pi$ is implied by inequality (iii) of proposition 3.

4. The higher-order terms

It was stated in § 2 that

$$j'''_{2n} = 9j_2 j'_{2n} + h_{2n} \quad (n \geqslant 2), \tag{4.1}$$

where h_{2n} is a polynomial in j_{2k}, f_{2k+1} $(k < n)$ and their derivatives. It will now be shown that the downstream behaviour of j_{2n} is determined by j_2.

Let us first consider the piece-wise solitary wave produced by a non-zero perturbation. Let $B_1 = \{h\}$ = the class of continuous functions satisfying

$$\sup_{\phi} \exp(2|\phi|)\,|h(\phi)| < \infty. \tag{4.2}$$

Note that j'_2 is a member of B_1.

Theorem 3. If $j_2(0) < -1$ and if h is in B_1, the differential equation

$$y'' = 9j_2 y + h \tag{4.3}$$

has a unique solution in B_1.

Proof. Let U_1 and U_2 be the two independent solutions to the homogeneous equation

$$y'' = 9j_2 y \tag{4.4}$$

such that $U_1 U'_2 - U_2 U'_1 = 1$ and $U_2 = -\frac{2}{27}j'_2$ for $\phi \leqslant 0$. Then if

$$\left.\begin{aligned} u_2(\phi) &= \tfrac{2}{3}\operatorname{sech}^2(\tfrac{3}{2}\phi - b)\tanh(\tfrac{3}{2}\phi - b),\\ u_1(\phi) &= \tfrac{3}{64}u_2(\phi)\sinh(6\phi - 4b) + \text{a calculable bounded function,} \end{aligned}\right\} \tag{4.5}$$

it can be shown that $U_1 = u_1$ and $U_2 = u_2$ for $\phi \leqslant 0$. Note that u_2 is in B_1, but u_1 is not. After the perturbation, j_2 is given by (3.26), i.e. its argument is translated by $\sigma = g + b$. Therefore

$$U_1(\phi) = a_1 u_1(\phi+\sigma) + a_2 u_2(\phi+\sigma), \quad U_2(\phi) = a_3 u_1(\phi+\sigma) + a_4 u_2(\phi+\sigma), \tag{4.6}$$

where the constants a_i satisfy $a_1 a_4 - a_2 a_3 = 1$.

It will now be shown that if y is in B_1, and if y is a solution to the homogeneous equation (4.4), then y is identically zero. First, note that for $\phi \leqslant 0$, y must be a multiple of u_2. We can therefore assume that $y = u_2$ for $\phi \leqslant 0$. Now set

$$W = \tfrac{2}{27}j'_2 + y. \tag{4.7}$$

Using the differential equation satisfied by j_2, (3.3), we have

$$W'' = 9j_2 W + \tfrac{2}{9}f, \quad W(0) = W'(0) = 0. \tag{4.8}$$

Therefore

$$W'(\phi) = 9\int_0^{\phi} j(s)\,W(s)\,ds + \tfrac{3}{2}F(\phi), \tag{4.9}$$

where $F' = f$. Since $F(A) = 0$, $W \geqslant 0$ implies that $W'(A) < 0$ (W cannot be identically zero since f is not). This implies that y is not in B_1, for if y is in B_1, then $y = cu(\phi_2 + \sigma)$ for $\phi \geqslant A$. Therefore $W(\phi) = (c - \frac{2}{9})\,u_2(\phi+\sigma)$, $\phi \geqslant A$, or

$$W(A) = -\tfrac{2}{27}(c - \tfrac{2}{9})j'_2(A), \quad W'(A) = -\tfrac{4}{81}(c - \tfrac{2}{9})j''_2(A). \tag{4.10}$$

Since $j_2(A) < -1$, $j_2''(A) > 0$. Furthermore $j_2' > 0$. Therefore $W(A)\,W'(A) > 0$. This contradicts the fact that $W \geqslant 0$ implies $W'(A) < 0$.

To complete the uniqueness argument, we must show that $W \geqslant 0$ on $[0, A]$. Let $k(\phi) = 9j_2(\phi) + 18$, $r = \sqrt{18}$. Then

$$W(\phi) = \frac{1}{r}\int_0^\phi \sin r(\phi - t)\, k(t)\, W(t)\, dt + \frac{2}{9r}\int_0^\phi \sin r(\phi - t) f(t)\, dt. \qquad (4.11)$$

Since $F(0) = 0$, the second integral can be integrated by parts to give

$$W(\phi) = \frac{1}{r}\int_0^\phi \sin r(\phi - t)\, k(t)\, W(t)\, dt + \frac{2}{9}\int_0^\phi \cos r(\phi - t)\, F(t)\, dt. \qquad (4.12)$$

It follows from the inequalities $rA < \frac{1}{2}\pi$ and $F \geqslant 0$ that this integral equation has a non-negative solution. This completes the uniqueness argument.

We return to the analysis of the inhomogeneous equation (4.3). Its solution can be written

$$y(\phi) = -U_1(\phi)\int_{-\infty}^{\phi} U_2(t)\, h(t)\, dt + U_2(\phi)\int_0^\phi U_1(t)\, h(t)\, dt + mU_2(\phi). \qquad (4.13)$$

It is easy to check that h in B_1 implies $\sup e^{-2\phi}\,|y(\phi)| < \infty$ for $\phi < 0$. The corresponding condition for positive ϕ will determine the constant m.

If $\phi > A$, then equations (4.6) can be substituted into (4.8). Using the equation $a_1 a_4 - a_2 a_3 = 1$, we get

$$y(\phi) = Mu_1(\phi + \sigma) + Nu_2(\phi + \sigma) + u_1(\phi + \sigma)\int_\phi^\infty u_2(t + \sigma)\, h(t)\, dt$$
$$+ u_2(\phi + \sigma)\int_A^\phi u_1(t + \sigma)\, h(t)\, dt, \qquad (4.14)$$

where

$$M = -a_1\int_{-\infty}^{A} U_2(t)\, h(t)\, dt + a_3\int_0^A U_1(t)\, h(t)\, dt - \int_A^\infty u_2(t + \sigma)\, h(t)\, dt + ma_3,$$
$$N = -a_2\int_{-\infty}^{A} U_2(t)\, h(t)\, dt + a_4\int_0^A U_1(t)\, h(t)\, dt.$$

Since $a_3 = 0$ violates the uniqueness argument, there is no difficulty in choosing m so that $M = 0$. This implies that y is in B_1, which was to be shown.

The characterization of j_{2n} in terms of j_2 when j_2 is periodic for $\phi > A$ is helped by a translation in the independent variable. The new origin is taken to be the second zero of $j_2'(\phi)$, $\phi \geqslant A$. With this new origin, we can say that j_2' is *locally odd*, that is

$$j_2'(-t) = -j_2'(t) \quad \text{for} \quad |t| \leqslant \tfrac{1}{2}T, \qquad (4.15)$$

where T is the period of j_2. The function class B_2 to which j_2' belongs can be defined as the class of continuous functions h such that

$$\left.\begin{array}{lll} \text{(i)} & \sup e^{2t}\,|h(t)| < \infty & (t < 0), \\ \text{(ii)} & h(t + T) = h(t) & (t > -\tfrac{1}{2}T), \\ \text{(iii)} & h \text{ is locally odd.} & \end{array}\right\} \qquad (4.16)$$

As in theorem 3, equation (4.1) has a unique solution in B_2 if h is in B_2. The proof uses the analysis of Littman (1957) of the unperturbed periodic flow. He showed that the solutions to the unperturbed homogeneous equations have the following properties: u_2 has period T but u_1 does not, and u_1 is locally even while u_2 is locally odd. The uniqueness argument is as in theorem 3. The existence argument selects the constant m so that y is periodic after the perturbation. Note that if y is in B, then

$$Y(\phi) = \int_{-\infty}^{\phi} y(t)\,dt$$

is periodic for $\phi > A$, and Y is bounded for all ϕ. If y is in B_1, we can also conclude that Y is bounded.

It has not yet been shown that j_2' in B_i implies that h_{2n} is in B_i, $i = 1, 2$. This follows from the fact that in the expansion of the function G in equation (2.11) of § 2, both θ_{2k+1} and τ_{2k}' are in B_i, and τ_{2k} is bounded.

5. The dependence of τ on f

Let τ be approximated along $\psi = 0$ by

$$\tau \cong a^2 j_2 + a^4 j_4, \tag{5.1}$$

and let $\tilde{\tau}$ be the unperturbed function. The difference between τ and $\tilde{\tau}$ will now be estimated. If we set $W_2 = j_2 - \tilde{j}_2$, then

$$W_2'' + 18W_2 = \tfrac{9}{2}(j_2 + \tilde{j}_2 + 4)\,W_2 + 3F_5, \quad W_2(0) = W_2'(0) = 0. \tag{5.2}$$

Using an integral equation similar to (4.11), we conclude that

$$|W_2| \leqslant \alpha\,|W_2| + \delta\,|3F_5|, \tag{5.3}$$

where
$$|h| = \max|h(\phi)| \quad (0 \leqslant \phi \leqslant A);$$
$$\alpha = 2(1-\cos rA), \quad \delta = A/r.$$

Therefore if $3rA < \pi$,

$$|W_2| \leqslant P_0\,|3F_5|, \quad P_0 = \delta/(1-\alpha). \tag{5.4}$$

This bound can be used in the integral equation satisfied by W' to give

$$|W_2'| \leqslant P_1\,|3F|, \quad P_1 = A(36P_0 + 1). \tag{5.5}$$

Using the bounds on W and W', we get

$$|W_2^{(k)}| \leqslant P_k\,\|3F_5\| \quad (0 \leqslant k \leqslant 5), \tag{5.6}$$

where
$$\|h\| = \max|h^{(m)}| \quad (1 \leqslant m \leqslant 5,\ P_k = \text{constant}).$$

Now let $W_4 = j_4' - \tilde{j}_4'$, then

$$W_4'' + 18W_4 = 9(j_2 + 2)\,W_4 + 9W_2 + h_4 - \tilde{h}_4. \tag{5.7}$$

Therefore
$$|W_4| \leqslant P_0(9P_0\,|3F| + |h_4 - \tilde{h}_4|). \tag{5.8}$$

It is straightforward to estimate $|h_4 - \tilde{h}_4|$ (see the explicit formula following (2.17)) to get

$$|W_4| \leqslant P_6 \max[\|f_5\|, |f_7|]. \tag{5.9}$$

Consequently
$$|\tau - \tilde{\tau}| \leqslant P_7\,\|f\|. \tag{5.10}$$

This estimate could be obtained for a higher-order approximation to τ, but this will not be done here. A standard argument can now be employed to prove the convergence of the iterative procedure of § 1.

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